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Collective phase synchronization in locally coupled limit-cycle oscillators

H. Hong,[1] Hyunggyu Park,[2] and M. Y. Choi[3,2]
[1]*Department of Physics, Chonbuk National University, Jeonju 561–756, Korea*
[2]*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*
[3]*Department of Physics, Seoul National University, Seoul 151-747, Korea*
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We study collective behavior of locally coupled limit-cycle oscillators with scattered intrinsic frequencies on d-dimensional lattices. A linear analysis shows that the system should always be desynchronized up to $d=4$. On the other hand, numerical investigation for $d=5$ and $d=6$ reveals the emergence of the synchronized (ordered) phase via a continuous transition from the fully random desynchronized phase. This demonstrates that the lower critical dimension for the phase synchronization in this system is $d_l=4$.

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Up to date, much attention has been paid to the collective behavior of coupled nonlinear oscillators since those systems of oscillators have been known to exhibit remarkable phenomena of synchronization [1]. Those phenomena have been observed in a number of physical, biological, and chemical systems, and understood rather well in terms of various model systems. In the system of globally-coupled oscillators such as the Kuramoto model [2,3], the mean-field (MF) theory is valid and yields analytic results to unveil the phase transition [2]. Systems of locally coupled oscillators, on the other hand, have not been studied much even though local coupling in the system is more realistic in nature. In some existing studies [4–6] collective synchronization, in particular frequency entrainment, has been investigated. However, even numerical results, as well as the analytic ones including heuristic arguments, do not provide a clear answer about the question of the lower critical dimension for the frequency entrainment. Phase synchronization has been also studied in the previous studies [7]; however, there are still many fundamental questions that are not answered.

In this Rapid Communication, we consider a system of locally coupled oscillators on d-dimensional lattices and use the relation with a typical model of growing surface, which allows a linear analysis to show the absence of synchronization up to $d=4$. On the other hand, numerical investigation performed for $d=5$ and 6 reveals the emergence of the synchronized (ordered) phase via a continuous transition, indicating the lower critical dimension for phase synchronization $d_l=4$.

We begin with the set of equations of motion governing the dynamics of N-coupled oscillators located at sites of a d-dimensional hypercubic lattice

$$\frac{d\phi_i}{dt} = \omega_i - K\sum_{j\in\Lambda_i} \sin(\phi_i - \phi_j), \tag{1}$$

where ϕ_i and ω_i stand for the phase and the intrinsic frequency of the ith oscillator $(i=1,2,\ldots,N)$, respectively. The intrinsic frequencies are assumed to be randomly distributed according to the Gaussian distribution function $g(\omega)$ with mean ω_0 and variance 2σ. For simplicity, we set $\omega_0\equiv 0$ without loss of generality. The second term on the right-hand side represents local interactions between the ith oscillator and its nearest neighbors the set of which is denoted by Λ_i.

Without any interaction $(K=0)$, each oscillator evolves with its own intrinsic frequency, resulting in that the system becomes trivially desynchronized. For $K>0$, the coupling term favors locally ordered (synchronized) states and competes against the randomizing force due to scattered intrinsic frequencies. When the coupling is strong enough to create globally ordered states, the system should exhibit collective synchronization. We here focus on phase synchronization which may be probed by the conventional phase order parameter

$$\Delta \equiv \left\langle \frac{1}{N}\left|\sum_{j=1}^{N} e^{i\phi_j}\right|\right\rangle, \tag{2}$$

where $\langle\cdots\rangle$ denotes the average over realizations of intrinsic frequencies. Phase synchronization is then identified by nonzero Δ in the thermodynamic limit.

Analytic results are available at the MF level. Namely, in the case of globally coupled oscillators, where each oscillator is coupled with every other one with equal strength K/N, it is known that phase synchronization emerges as $\Delta\sim(K-K_c)^{\beta}$ with $\beta=1/2$ near the critical coupling strength $K_c=2/\pi g(0)$ [2] while the correlation length diverges as $\xi\sim|K-K_c|^{-\nu}$ with $\nu=1/2$ [8].

When the oscillators are locally coupled, the system has been little investigated. Since the nonlinear nature of the sine coupling term in Eq. (1) is the major obstacle toward analytic treatment, we first suppose that, for sufficiently strong-coupling strength K, the phase difference between any nearest neighboring oscillators is small enough to allow the expansion of the sine function in the linear regime. With the appropriate continuum limit taken in space, the linearized evolution equation for the phase $\phi(\mathbf{x},t)$ reads

$$\frac{\partial\phi}{\partial t} = \omega(\mathbf{x}) + K\nabla^2\phi + O(\nabla^4\phi), \tag{3}$$

where $\omega(\mathbf{x})$ are uncorrelated random variables, satisfying $\langle\omega(\mathbf{x})\rangle=0$ and $\langle\omega(\mathbf{x})\omega(\mathbf{x}')\rangle=2\sigma\delta(\mathbf{x}-\mathbf{x}')$ [9]. We also relax the constraint $0 \leqslant \phi < 2\pi$ and extend the range of ϕ to $(-\infty, \infty)$, for convenience.

With the irrelevant high order terms neglected, this equation reminds us of the celebrated Edwards-Wilkinson (EW) equation [10], traditionally describing certain surface evolution, by interpreting the phase $\phi(\mathbf{x},t)$ as the front height of the growing surface. Note, however, that the noise $\omega(\mathbf{x})$ is generated not by conventional spatio-temporal disorder but by so-called columnar disorder (with spatial dependence only).

In the context of surface growth models, a central quantity of interest is the surface fluctuation width W defined by

$$W^2(t) = \frac{1}{L^d}\int^L d^d\mathbf{x}\langle[\phi(\mathbf{x},t) - \overline{\phi}(t)]^2\rangle, \qquad (4)$$

where L is the linear size of the d-dimensional lattice ($L^d = N$) and $\overline{\phi}(t)$ the spatial average of the phase $\phi(\mathbf{x},t)$. By means of the Fourier transforms, one can easily solve Eq. (3) to find in the long time limit ($Kt \gg L^2$) that the steady-state surface width scales for large L [8]

$$\begin{aligned} W^2 &\sim (2\sigma/K^2)L^{4-d}, \quad d<4 \\ &\simeq (\sigma/4\pi^2K^2)\ln L, \quad d=4 \\ &\sim 2\sigma/K^2, \quad d>4. \end{aligned} \qquad (5)$$

At any finite values of K, the surface width W thus diverges as $L \to \infty$ for $d \leqslant 4$ whereas it remains finite for $d>4$. This indicates that the surface is always rough (except at $K=\infty$) for $d \leqslant 4$ and always smooth (except at $K=0$) for $d>4$.

It is also straightforward to derive the steady-state probability distribution [8]

$$P[\{\phi\}] \sim \exp\left[-(K^2/4\sigma)\int(\nabla^2\phi)^2 d^d\mathbf{x}\right]. \qquad (6)$$

Notice that the Gaussian property of the probability distribution links W analytically to the phase order parameter via $\Delta = \exp[-W^2/2]$. Therefore our results for W, translated into the phase synchronization language, show that the oscillators are always desynchronized ($\Delta=0$) for $d \leqslant 4$ and always synchronized ($\Delta \neq 0$) for $d>4$ in this linearized model.

Our linear theory is valid in the strong-coupling regime; as the weak-coupling regime is approached, the original (nonlinear) system should be more disordered than the prediction of the linear theory. This establishes that the full nonlinear system described by Eq. (1) should also be desynchronized for $d \leqslant 4$ at any finite K. For $d>4$, it is reasonable to expect a phase synchronization (roughening) transition at a finite value of K, although one may not exclude the possibility of either the full destruction of the synchronized phase at any finite K or the absence of the desynchronized phase at any nonzero K.

Before investigating the full nonlinear system described by Eq. (1), we consider another standard quantity in surface growth models, the height-height correlation function $C(\mathbf{x},t) \equiv \langle[\phi(\mathbf{x},t) - \phi(\mathbf{0},t)]^2\rangle$. In the linearized regime governed by Eq. (3), we find the steady-state behavior for small $x \equiv |\mathbf{x}|$ [8]

$$\begin{aligned} C(x) &\sim (2\sigma/K^2)x^2L^{2-d}, \quad d<2 \\ &\simeq (\sigma/2\pi K^2)x^2\ln L, \quad d=2 \\ &\sim (2\sigma/K^2)x^{4-d}, \quad d>2. \end{aligned} \qquad (7)$$

Note that for $d \leqslant 2$ the correlation $C(x)$ diverges with system size L, which implies that the average nearest neighbor phase (height) difference $G = \langle(\nabla\phi)^2\rangle^{1/2}$ is unbounded for any finite K in the thermodynamic limit. As our linear theory is based on the boundedness of $|\nabla\phi|$, there is no range of K where the linear theory applies for $d \leqslant 2$. In contrast, for $d>2$, G is finite and the linear theory is self-consistent at least for large K where $G(K) \lesssim \mathcal{O}(1)$. We now examine the nonlinear effects due to the sine coupling in Eq. (1). Unlike in the linearized case, phase ϕ may not be bounded even in a finite system but diverge eventually with a finite angular velocity, once its intrinsic-frequency term wins over the nearest-neighbor coupling term. In the weak-coupling regime (for small K), these *runaway* oscillators with scattered angular velocities dominate, and their phases become completely random to one another, leading to the behavior $\Delta \sim N^{-1/2} = L^{-d/2}$. On the other hand, in the strong-coupling regime where the linear theory applies, Δ vanishes exponentially for $d=3$ and algebraically for $d=4$, with an exponent depending on K [see Eq. (5)].

We integrate numerically Eq. (1) and measure the phase order parameter at various values of K and L for $d=2$ to 6. For convenience, periodic boundary conditions have been employed and 2σ has been set equal to unity. We start from the uniform initial condition ($\phi_i=0$) for a given set of $\{\omega_i\}$, chosen randomly according to the Gaussian distribution $g(\omega) \sim \exp(-\omega^2/4\sigma)$, and measure the order parameter Δ averaged over the data in the steady state, reached after appropriate transient time ($Kt \gg L^2$). Here we have used Heun's method [11] to integrate up to 4×10^4 time steps, with the time step $\delta t=0.05$, and also average over 100 independent sets of $\{\omega_i\}$. Figure 1 displays the numerical results for the order parameter. For $d=2$ and 3, it is clearly observed that the order parameter decreases rapidly with the system size and seemingly approaches zero in the thermodynamic limit for any finite K. Detailed finite-size analysis [8] shows $\Delta \sim L^{-d/2}$ in the weak-coupling regime, implying that phases are completely random and the system is dominated by runaway oscillators. For $d=2$, this fully random phase extends to the regime of large K, while for $d=3$ the linear theory predicting correlated phases [see Eq. (7)] appears to work for large K, namely, the data fit well to $\Delta \sim \exp[-(\sigma/4\pi^3K^2)L]$ for $K>K_0$ with $G(K_0) \approx \mathcal{O}(1)$. Numerically, we find that $K_0 \approx \sqrt{2\sigma/\pi}$ [12]. The data for $d=4$ seem to suggest that for large K, Δ remains finite even in the thermodynamic limit, which contradicts our prediction based on the linear analysis. To resolve this puzzle, we analyze our data carefully by means of finite-size scaling, and show in Fig. 2 the log-log plots of Δ versus L^{-1} at various values of K. Manifested for $K \lesssim 0.28$ is the fully random phase $\Delta \sim L^{-2}$. For $K \gtrsim 0.40$, Δ still decreases algebraically with L (see the inset of Fig. 2) $\Delta \sim L^{-\delta(K)}$. It is pleasing that our data for $K \gtrsim 0.40$ agree perfectly with the prediction of the linear theory, $\delta(K) = \sigma/8\pi^2K^2$ from Eq. (5). This result confirms that there is no

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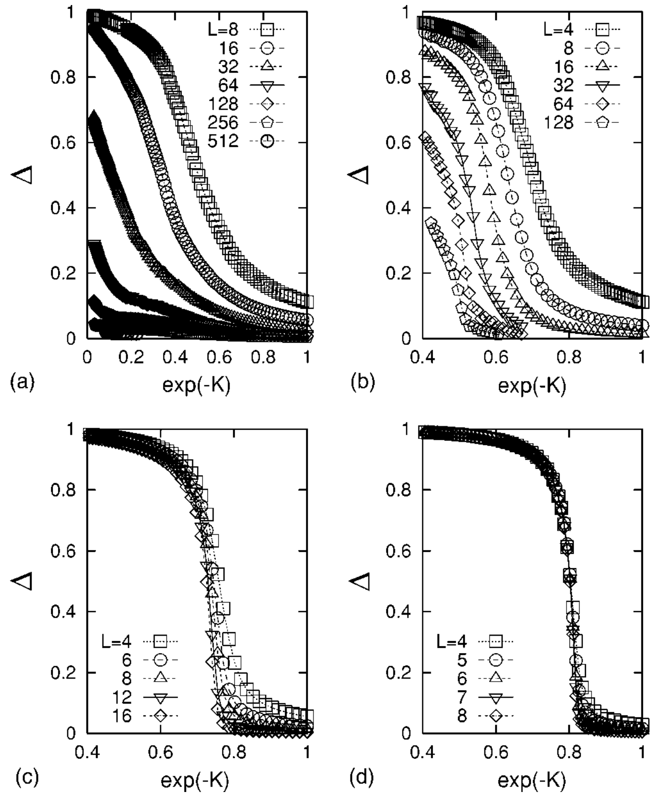

FIG. 1. Behavior of the order parameter Δ with the coupling strength K, plotted in terms of $\exp(-K)$, in systems of various size L for (a) $d=2$, (b) $d=3$, (c) $d=4$, and (d) $d=5$. Symbol sizes correspond to statistical errors of the data.

synchronized phase at any finite K for $d=4$. It would be interesting to explore the possibility of a phase transition near $K \approx K_0 = \sqrt{\sigma/4}$ between the fully random phase and the critical phase described by the linear theory; this is currently under investigation. For $d=5$, it looks evident that there exists an ordered (synchronized) phase extended to finite values of K. Similarly to the $d=4$ case, the log-log plots of Δ versus L^{-1} are drawn in Fig. 3. For $K \lesssim 0.19$, we find the fully random phase $\Delta \sim L^{-5/2}$. For $K \gtrsim 0.21$, on the other hand, Δ, first decreasing slightly with L, eventually con-

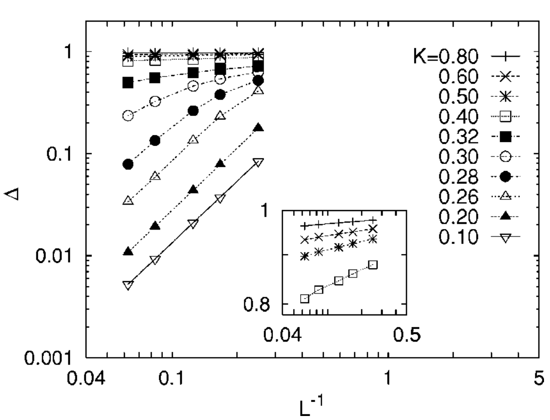

FIG. 2. Log-log plots of Δ versus L^{-1} for $d=4$ at various values of K. The data for large K are shown in the inset for better visibility. Lines are merely guides for eyes.

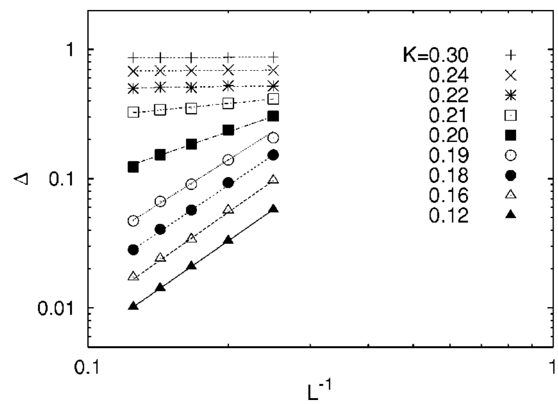

FIG. 3. Log-log plots of Δ versus L^{-1} for $d=5$ at various values of K. Lines are merely guides for eyes.

verges to a nonzero value. In fact, for $K \gtrsim 0.24$, this saturated value coincides perfectly well with the linear-theory value: $\Delta = \exp[-\sigma/12\pi^2 K^2]$. Note here that the linear theory breaks down for $K \lesssim K_0 = \sqrt{\sigma/9} \approx 0.24$ and the transition into the fully random phase apparently occurs a little later at $K_c \approx 0.20$. It may be very interesting to understand this phase transition from the stability analysis in the weak-coupling limit.

We next study the critical behavior near the synchronization transition. In a finite system, we assume the finite-size scaling relation

$$\Delta = L^{-\beta/\nu} f[(K-K_c)L^{1/\nu}], \tag{8}$$

where the scaling function behaves $f(x) \sim x^{\beta}$ as $x \to +\infty$ and $f(x) \sim \text{const}$ as $x \to 0$. At criticality, it leads to

$$\Delta(K_c, L) \sim L^{-\beta/\nu}. \tag{9}$$

To estimate efficiently the exponent β/ν and the transition point K_c, we introduce the effective exponent

$$\beta/\nu(L) = -\ln[\Delta(L')/\Delta(L)]/\ln(L'/L), \tag{10}$$

which is expected to approach zero, β/ν, and $d/2$ for $K > K_c$, $K = K_c$, and $K < K_c$, respectively, as $L \to \infty$.

The effective exponent for $d=5$, computed at various values of K, is plotted in Fig. 4. The data for $K \leqslant 0.19$ apparently converge to the weak-coupling value $5/2$, while those for $K \geqslant 0.21$ converge to zero within statistical errors. Only the data at $K=0.20$ appear to converge to a nontrivial value. We thus estimate the critical coupling strength $K_c = 0.200(5)$ and the exponent ratio $\beta/\nu = 1.6(3)$.

To check the finite-size scaling relation directly, we plot $\Delta L^{\beta/\nu}$ versus $(K/K_c - 1)L^{1/\nu}$ in Fig. 5 and find that the data for various values of L and K are best collapsed to a curve with choices of $K_c = 0.200(5)$, $\beta/\nu = 1.4(3)$, and $\nu = 0.45(10)$, which results in $\beta = 0.63(20)$. As expected, the resulting scaling function $f(x)$ converges to a constant for small x, and diverges as x^{β} for large x (see Fig. 5).

We summarize our results for $d=5$: $\beta/\nu = 1.5(3)$, $\nu = 0.45(10)$, $K_c = 0.200(5)$. Note the apparently substantial deviations from the MF values, $\beta/\nu = 1$ and $\nu = 1/2$, although the latter may not be totally excluded. In view of the argu-

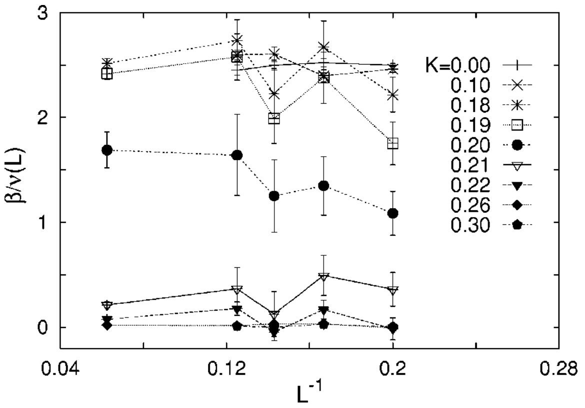

FIG. 4. Effective exponent $\beta/\nu(L)$ versus L^{-1} for $d=5$ at various values of K.

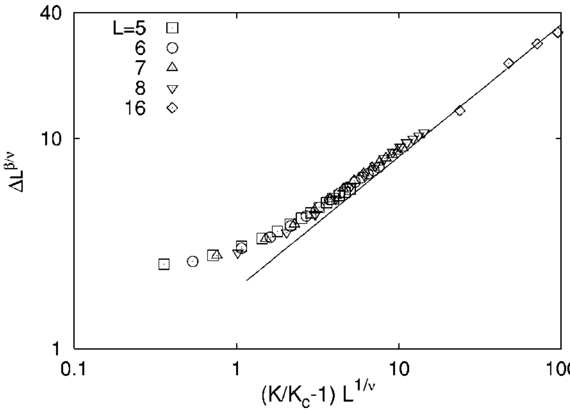

FIG. 5. Data collapse of $\Delta L^{\beta/\nu}$ against $(K/K_c-1)L^{1/\nu}$ in the log-log scale for various values of the system size and coupling strength. The best collapse is achieved with $\beta/\nu=1.4(3)$ and $\nu=0.45(10)$. The straight line has the slope 0.63, giving an estimation of β.

ment for the MF nature [8], these apparent deviations are rather unexpected and their origin is unclear at this stage. Similarly, we find for $d=6$, $\beta/\nu=1.0(3)$, $\nu=0.45(10)$, and $K_c=0.158(5)$, which seem to be consistent with the MF values.

In summary, we have explored the phase synchronization phenomena in the system of locally coupled oscillators with scattered intrinsic frequencies on d-dimensional lattices. A linear analysis shows that the strong-coupling regime can be described by the EW surface growth equation with columnar disorder for $d\geqslant 3$. It has been shown analytically that the system is always desynchronized up to $d=4$, while numerical integration for $d\geqslant 5$ has demonstrated the emergence of the synchronized (ordered) phase via a continuous transition from the desynchronized phase. The lower critical dimension for phase synchronization is thus given by $d_l=4$, but the critical behavior explored for $d=5$ and 6 does not give a conclusive result for the upper critical dimension.

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